An Investigation of Equivalence between Bulk-based and Brane-based Approaches

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There are two different approaches to handle brane-world models brane-based or bulk-based. In the brane-based approach, the brane is chosen to be fixed on a coordinate system, whereas in the bulk-based approach, it is no longer static as it moves along the extra dimension. It is aimed to get a general formalism of the equivalence between two approaches obtained for a specific model in Mukohyama et all [12]. We found that calculations driven by a general anisotropic bulk-based metric yields a brane-based metric in Gaussian Normal Coordinates by conserving spatial anisotropy.

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I. INTRODUCTION

The discovery of Rundall and Sundrum (RS) [1, 2] opened a new window to our conventional cosmology. Their models give an alternative solution of hierarchy problem in which the additional dimension does not have to be small as in Kaluza-Klein model. The extra dimension can be infinite but warped. Our observable universe could be a 1+3 surface (called a brane) embedded 5-dimensional spacetime (called a bulk). Standard Model matters and fields are confined on the brane while gravity is permitted to propagate the bulk.

RS Models and subsequent generalization from a Minkowski brane to a Friedmann-Robertson-Walker (FRW) brane [3, 4, 5, 6, 7, 8, 9] were derived as solutions in particular coordinates of the 5-dimensional Einstein equations, together with the junction conditions at the Z2-symmetric brane.

$$R^5_{AB} - \frac{1}{2}g^5_{AB} = -\Lambda_5 g^5_{AB} + \kappa_5^2 T^5_{AB} \tag{1}$$

where T^5_{AB} represents any 5-dimensional energy-momentum of the gravitational sector and it provides a conservation equation

$$\nabla_A T^{AB} = 0 \tag{2}$$

The 5-dimensional curvature tensor has Weyl and Ricci parts

$$\begin{aligned} R^5_{ABCD} \;&=\; C^5_{ABCD} + \frac{1}{3}[R^5_{AC}g^5_{BD} - R^5_{AD}g^5_{BC} + R^5_{BD}g^5_{AC} - R^5_{BC}g^5_{AD}] \\ &\quad -\frac{1}{12}R^5[g^5_{AC}g^5_{BD} - g^5_{AD}g^5_{BC}] \end{aligned} \tag{3}$$

where C^5_{ABCD} is Weyl Tensor.

Another way of deriving the brane world cosmological equations is to use induced equations on the brane. Shiromizu-Maeda-Sasaki [10] obtained 4-dimensional induced field equations by projecting 5-dimensional equations on the brane and using Israel Junction Conditions.

$$G^4_{AB} = -\Lambda_4 h_{AB} + 8\pi G_5 \tau_{AB} + \kappa_5^4 \pi^5_{AB} - E^5_{AB} \tag{4}$$

$$\Lambda_4 = \frac{1}{2}\kappa_5^2[\Lambda_5 + \frac{1}{2}\kappa_5^2\lambda^2] \tag{5}$$

$$G_5 = \frac{\kappa_5^4\lambda}{48\pi} \tag{6}$$

$$\pi^5_{AB} = -\frac{1}{4}\tau_{AC}\tau_B{}^C + \frac{1}{12}\tau\tau_{AB} + \frac{1}{8}h_{AB}\tau_{CD}\tau^{CD} - \frac{1}{24}h_{AB}\tau^2 \tag{7}$$

These equations are also important to give us the connection with four and five dimensional quantities. They also contain bulk effects on the brane. In our previous study, [11] when investigating an anisotropic model, four dimensional induced field equations were employed. But in this study, the calculations are derived by using five dimensional equations (1).

Except from these two ways of cosmological model construction, there are also two more approaches: brane-based or bulk based. In the brane-based approach, the brane is chosen to be fixed on a coordinate system. The 5d metric depends on both time and extra coordinate. On the contrary, in the bulk-based approach the brane is no longer static as it moves along the extra dimension and the bulk metric has static form. It is shown that both approaches are equivalent when homogeneity and isotropy are considered [12]. There is no existing study showing what kind of equality will be obtained in the manner of different models, especially anisotropic models which we are interested in. The main purpose of the study is to generalizing the formulation for the afterwards getting the corresponding versions in brane-based approach.

We first construct formulas providing the transformation of the bulk-based approach to the brane-based one in Sec.2. It is shown that our results cover the solutions in [12] in which the equivalence metric of Scw-AdS spacetime has been explored. At the end of the driven calculations, we find that the equivalence of the most general anisotropic bulk-based metric is an anisotropic metric in the form of Gaussian Normal Coordinates as expected.

Because the Gaussian Normal Coordinates are the most operational coordinates which allow to induce the metric from 5-dimensional spacetime to 4-dimensional hypersurface, the final conclusion of the study is important for obtaining correspondence of any models in Gaussian Normal Coordinates. Apart from constituting general formalism, we get the solutions for a bulk-based model named Gergely-Maartens metric, and construct corresponding brane-based solution as an application in Sec.3.

II. GENERAL TRANSFORMATION FROM BULK-BASED METRIC TO BRANE-BASED ONE

The most general 5-dimensional anisotropic bulk-based metric admitting non-zero Killing vector space is

$$ds^2 = -A_0(\hat{r})d\hat{t}^2 + A_{ij}(\hat{r})d\hat{x}^i d\hat{x}^j + A_4(\hat{r})d\hat{r}^2 \tag{8}$$

where $\hat{r}$ denotes the extra spatial coordinate and $i,j = 1,2,3$ are indices of 3-dimensional spacetime. The 4-dimensional brane corresponding our universe moves along the extra dimension and is described by (τ, x^i) coordinates. Base vectors and one forms in the 5-dimensional and 4-dimensional spacetime respectively are in below.

$$e_A \equiv \partial_A = \frac{\partial}{\partial \hat{x}_A} = (\partial_{\hat{t}}, \partial_{\hat{x}^i}, \partial_{\hat{r}}) \qquad \omega^A \equiv d\hat{x}^A = (d\hat{t}, d\hat{x}^i, d\hat{r}) \tag{9}$$

$$e_\mu \equiv \partial_\mu = \frac{\partial}{\partial \hat{x}_\mu} = (\partial_\tau, \partial_{x^i}) \qquad \omega^\mu \equiv dx^\mu = (d\tau, dx^i) \tag{10}$$

Here we define $A = 0..4$, $\mu = 0..3$. The brane represented by $\hat{r} = R(\hat{t})$ hypersurfaces can be induced on 4-dimensional spacetime via following transformations.

$$\begin{gathered} \hat{t} = T(\tau) \rightarrow d\hat{t} = \dot{T}d\tau \\ \hat{x}^i = x^i \rightarrow d\hat{x}^i = dx^i \\ \hat{r} = R(\tau) \rightarrow d\hat{r} = \dot{R}d\tau \end{gathered} \tag{11}$$

The induced metric on brane is then

$$ds^2_{brane} = -(A_0\dot{T}^2 - A_4\dot{R}^2)d\tau^2 + A_{ij}(R(\tau))dx^i dx^j \tag{12}$$

where we introduced cosmological time τ and cosmological scale factor $R(\tau)$. The dot denotes to derivative respect to τ. Now we can construct a vector space generating by tangent vectors of geodesics which intersect with hypersurface $\hat{r} = R(\hat{t})$ perpendicularly

$$u^A = e^A_\tau \partial_A = \dot{T}\partial_{\hat{t}} + \dot{R}\partial_{\hat{r}} \tag{13}$$

We choose geodesics as spacelike and have zero $\hat{x}^i$-components to provide a timelike hypersurface. The Killing field of bulk spacetime helps us to find constants of motion along geodesics.

$$g_{AB}u^A\xi^B = -E \tag{14}$$

$$g_{AB}u^A u^B = 1 \tag{15}$$

where E is an integration constant. Using tangent vector's components in (13), we obtain

$$u^A = \left(\frac{E}{A_0}, 0, 0, 0, \mp\frac{A_0 + E^2}{A_0 A_4}\right) \tag{16}$$

The trajectory of the geodesic is given by

$$\frac{dx^A}{dw} = u^A \tag{17}$$

here w is the affine parameter. Every points P on the hypersurface described by (τ, x^i) coordinates intersect perpendicularly with an affinely parameterized geodesic. Hence we can describe P via a new coordinate set (τ, x^i, w), where the new coordinate w is now an extra spatial coordinate of P and this system is called by brane-based coordinates. One can easily construct brane-based metric from bulk-based one by applying transformations: $\hat{r} = \hat{r}(\tau, w)$, $\hat{t} = \hat{t}(\tau, w)$

$$d\hat{t} = \left(\frac{\partial \hat{t}}{\partial \tau}\right) d\tau + \left(\frac{\partial \hat{t}}{\partial w}\right) dw = e^{\hat{t}}_{\tau} d\tau + e^{\hat{t}}_{w} dw \tag{18}$$

$$d\hat{r} = \left(\frac{\partial \hat{r}}{\partial \tau}\right) d\tau + \left(\frac{\partial \hat{r}}{\partial w}\right) dw = e^{\hat{r}}_{\tau} d\tau + e^{\hat{r}}_{w} dw \tag{19}$$

substituting them in (8) gives

$$\begin{aligned} ds^2 \;=\; & -\left[A_0\left(\frac{\partial \hat{t}}{\partial \tau}\right)^2 - A_4\left(\frac{\partial \hat{r}}{\partial \tau}\right)^2\right] d\tau^2 + A_{ij}dx^i dx^j \\ & + \left[-A_0\left(\frac{\partial \hat{t}}{\partial w}\right)^2 + A_4\left(\frac{\partial \hat{r}}{\partial w}\right)^2\right] dw^2 \\ & +2\left[-A_0\left(\frac{\partial \hat{t}}{\partial \tau}\right)\left(\frac{\partial \hat{t}}{\partial w}\right) + A_4\left(\frac{\partial \hat{r}}{\partial \tau}\right)\left(\frac{\partial \hat{r}}{\partial w}\right)\right] d\tau dw \end{aligned} \tag{20}$$

The final metric in (20) is a general form of brane-based metric which is transformed from bulk-based metric and here w is the affine parameter. We need to find the exact forms of transformation coefficients denoted by partial derivatives in (20). Replacing (16) in (17) we get two equations

$$u^{\hat{t}} = e^{\hat{t}}_{w} = \frac{\partial \hat{t}}{\partial w} = \frac{E}{A_0} \tag{21}$$

$$u^{\hat{r}} = e^{\hat{r}}_{w} = \frac{\partial \hat{r}}{\partial w} = \mp\sqrt{\frac{A_0 + E^2}{A_0 A_4}} \tag{22}$$

in which the latter one gives a integral relation between extra coordinates of two approaches. In the case of components of bulk-based metric are known, (26) can be solved exactly.

$$\mp w + w_0(\tau) = \int \frac{d\hat{r}}{\sqrt{\frac{A_0+E^2}{A_0 A_4}}} \tag{23}$$

On the other hand, we get transverse coefficients in (21-22) from $dw/dx^B = g_{AB}u^A$

$$e_{\hat{t}w} = \frac{\partial w}{\partial \hat{t}} = -E \tag{24}$$

$$e_{\hat{x}^i w} = \frac{\partial w}{\partial \hat{x}^i} = 0 \tag{25}$$

$$e_{\hat{r}w} = \frac{\partial w}{\partial \hat{r}} = \pm\sqrt{\frac{A_4\left(A_0+E^2\right)}{A_0}} \tag{26}$$

The integrability condition $ddw = 0$ is equivalent to

$$\left(\frac{\partial w}{\partial \hat{t}}\right) d\hat{t} = -\left(\frac{\partial w}{\partial \hat{r}}\right) d\hat{r} \tag{27}$$

and gives a ratio of coefficients

$$\left(\frac{\partial \hat{t}}{\partial \tau}\right) / \left(\frac{\partial \hat{r}}{\partial \tau}\right) = -\left(\frac{\partial w}{\partial \hat{r}}\right) / \left(\frac{\partial w}{\partial \hat{t}}\right) = \pm\sqrt{\frac{A_4\left(A_0+E^2\right)}{A_0 E^2}} \tag{28}$$

The transverse ratio is then

$$\left(\frac{\partial \tau}{\partial \hat{t}}\right) / \left(\frac{\partial \tau}{\partial \hat{r}}\right) = -\left(\frac{\partial \hat{r}}{\partial w}\right) / \left(\frac{\partial \hat{t}}{\partial w}\right) = \pm\sqrt{\frac{A_0\left(A_0+E^2\right)}{A_4 E^2}} \tag{29}$$

In order to see all transformation coefficients which we have calculated by now let us put them into a matrix form as below

$$(\hat{t},\hat{x}^i,\hat{r}) \rightarrow (\tau, x^i, w): \begin{pmatrix} e^{\hat{t}}_{\tau} & e^{\hat{t}}_{x^i} & e^{\hat{t}}_{w} \\ e^{\hat{x}^i}_{\tau} & e^{\hat{x}^i}_{x^i} & e^{\hat{x}^i}_{w} \\ e^{\hat{r}}_{\tau} & e^{\hat{r}}_{x^i} & e^{\hat{r}}_{w} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & \frac{E}{A_0} \\ 0 & 1 & 0 \\ a_{31} & 0 & \pm\sqrt{\frac{(A_0+E^2)}{A_4 A_0}} \end{pmatrix} \tag{30}$$

$$(\tau, x^i, w) \rightarrow (\hat{t},\hat{x}^i,\hat{r}): \begin{pmatrix} e_{\hat{t}\tau} & e_{\hat{t}x^i} & e_{\hat{t}w} \\ e_{\hat{x}^i\tau} & e_{\hat{x}^i x^i} & e_{\hat{x}^i w} \\ e_{\hat{r}\tau} & e_{\hat{r}x^i} & e_{\hat{r}w} \end{pmatrix} = \begin{pmatrix} a^{11} & 0 & -E \\ 0 & 1 & 0 \\ a^{31} & 0 & \pm\sqrt{\frac{A_4(A_0+E^2)}{A_0}} \end{pmatrix} \tag{31}$$

Note that two of the matrix components remain unknown, but they depend on each other according to these equations

$$e^{\hat{t}}_{\tau}/e^{\hat{r}}_{\tau} = a_{11}/a_{31} = \pm\sqrt{\frac{A_4\left(A_0+E^2\right)}{A_0 E^2}} \tag{32}$$

$$e_{\hat{t}\tau}/e_{\hat{r}\tau} = a^{11}/a^{31} = \pm\sqrt{\frac{A_0\left(A_0+E^2\right)}{A_4 E^2}} \tag{33}$$

Substituting partial derivatives from above matrix in (20), we find final form of brane-based metric

$$ds^2 = -\frac{A_0(\hat{r}(\tau,w))A_4(\hat{r}(\tau,w))}{E^2}\left(\frac{\partial \hat{r}}{\partial \tau}\right)^2 d\tau^2 \; + A_{ij}(\hat{r}(\tau,w))dx^i dx^j + dw^2 \tag{34}$$

which is a general form of brane-based metric in Gaussian Normal Coordinates but contains spatial anisotropy as expected. If we choose metric components as $A_0 = f(\hat{r})$, $A_4 = 1/f(\hat{r})$ and isotropic 3-dimensional metric, Eq. (34) readily yields the result in Ref. [12].

All terms in (34) depend on brane coordinates (τ, w). The motion constant E of bulk geodesics would not be a constant anymore on the brane and turns out to be $E = E(\tau)$. Besides that the term showing partial derivative can be calculated from integral relation (23) after obtaining explicit forms of metric components. For this purpose one must solve vacuum Einstein Field Equations in the bulk and replace them with transformed metric.

The general form of brane-based metric for anisotropic model given by (34) produce the induced metric on brane by setting $w = 0$. The result metric represents 4-dimensional spacetime which determined by Egs.(4-7).

III. TRANSFORMATION FOR GERGELY-MAARTENS METRIC

In this section we present an application regarding the transformation introduced above. Gergely-Maartens (GM) metric [13] can be taken as static 5d bulk-based metric for starters. This metric corresponds a Non-SchAds bulk spacetime with static Friedman brane.

$$\Gamma^2 ds^2 = -F^2(\hat{r},\epsilon)d\hat{t}^2 + d\hat{\chi}^2 + \mathcal{H}\left(\hat{\chi};\epsilon\right)\left(d\hat{\theta}^2 + \sin^2\hat{\theta}d\hat{\phi}^2\right) + d\hat{r}^2 \tag{35}$$

$$\mathcal{H}\left(\hat{\chi};k\right) = \begin{cases} \sin\hat{\chi}\ , & \epsilon = 1 \\ \hat{\chi}\ , & \epsilon = 0 \\ \sinh\hat{\chi}\ , & \epsilon = -1 \end{cases} . \tag{36}$$

where $\Gamma; \kappa^2\Lambda_5 = 3\epsilon\Gamma^2$ gives the magnitude of the cosmological constant and ϵ its sign. If $\epsilon = 0$, then Γ is a removable constant.

$$G_{AB} = -\kappa^2\Lambda_5 g_{AB} \tag{37}$$

By solving (35) from the 5-dimensional vacuum Einstein equation, we find

$$F(\hat{r},\epsilon) = \begin{cases} A\cos(\sqrt{2}\hat{r}) + B\sin(\sqrt{2}\hat{r}), & \epsilon = 1 \\ (A + B\sqrt{2}\hat{r})\ , & \epsilon = 0 \\ A\cosh(\sqrt{2}\hat{r}) + B\sinh(\sqrt{2}\hat{r})\ , & \epsilon = -1 \end{cases} . \tag{38}$$

here A and B are constants.

To get corresponding brane-based metric of GM, we first have to solve the integration in (23) by substituting $A_0 = F^2(\hat{r},\epsilon), A_4 = 1$

$$\mp w + w_0(\tau) = \int \frac{d\hat{r}}{\sqrt{\frac{F^2+E^2}{F^2}}} \tag{39}$$

Solutions are listed below for each case of ϵ

$$\mp w + w_0(\tau) = \begin{cases} \frac{1}{2\sqrt{2}}\arcsin\left[\frac{\sqrt{a^2x^2+cx+b^2}}{(c^2-4ab)|x|}2i(ax-b)(2ab-c)\right], & \epsilon = 1 \\ \frac{\sqrt{(A+B\sqrt{2}\hat{r})^2\ +E^2}}{\sqrt{2}B}, & \epsilon = 0 \\ \frac{1}{2\sqrt{2}}\arcsin\left[\frac{\sqrt{\alpha^2y^2+\beta y+\gamma^2}}{(4\alpha\gamma-\beta^2)|y|}2i(\alpha y-\gamma)(2\alpha\gamma-\beta)\right]\ , & \epsilon = -1 \end{cases} \tag{40}$$

Here we defined $x = \exp(i2\sqrt{2}\hat{r}), a = \frac{A}{2} + \frac{B}{2i}, b = \frac{A}{2} - \frac{B}{2i}, c = 2ab + E^2$ and $y = \exp(2\sqrt{2}\hat{r}), \alpha = \frac{A}{2} + \frac{B}{2}, \gamma = \frac{A}{2} - \frac{B}{2}, \beta = 2\alpha\gamma + E^2$

We show explicit calculation for the $\epsilon = 0$ case only, but the other cases can also be handled by the same calculations.

$$\left(A + B\sqrt{2}\hat{r}\right)^2 + E^2 = 2B^2\left(\mp w + w_0\right)^2 \tag{41}$$

To determine the constants E and w_0, we use the fact that the geodesic intersects with the hypersurface $\hat{r} = R(\hat{t})$ perpendicularly at $\hat{t} = \hat{t_0}$ and that the affine parameter w is zero on the hypersurface.

$$\left(A + B\sqrt{2}R\right)^2 + E^2 = 2B^2 w_0^2 \tag{42}$$

Substituting w_0 into (41) yields

$$\hat{r} = \frac{1}{\sqrt{2}B}\left\{\left(\mp\sqrt{2}Bw + \sqrt{A + B\sqrt{2}R + E^2}\right)^2 - E^2\right\}^{1/2} - \frac{A}{\sqrt{2}B} \tag{43}$$

Now we can derive brane-based metric of (35) as following

$$ds^2 = -\Psi(\tau, w)d\tau^2 + d\chi^2 + \mathcal{H}(\chi;\epsilon)\left(d\theta^2 + \sin^2\theta d\phi^2\right) + dw^2 \tag{44}$$

where

$$\Psi(\tau, w) = \left\{\frac{(A + B\sqrt{2}R)}{HR}\frac{(\mp w + w_0)}{w_0} + \left(\frac{\dot{H}}{H} + H\right)\frac{\mp w}{\sqrt{2}Bw_0}\right\}\sqrt{2B^2\left(\mp w + w_0\right)^2 - H^2R^2} \tag{45}$$

and $H = \dot{R}/R$ is the Hubble constant. If one sets the $w = 0$ in (45), it could be shown that 4-dimensional metric is equivalent to Kantoski-Sach spacetime [14].

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